

Prior learning

Area of a parallelogram
 $A = bh$
Area of a triangle
 $A = \frac{1}{2}(bh)$
Area of a trapezoid
 $A = \frac{1}{2}(a + b)h$
Area of a circle
 $A = \pi r^2$
Circumference of a circle
 $C = 2\pi r$
Volume of a cuboid
 $V = lwh$
Volume of a cylinder
 $V = \pi r^2 h$
Volume of a prism
 $V = Ah$
Area of the curved surface of a cylinder
 $A = 2\pi rh$

Distance between two points (x_1, y_1) and (x_2, y_2)
 $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Midpoint of a line segment
 $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

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Gradients of perpendicular lines
 $m_1 m_2 = -1$

Number and algebra

n th term of an arithmetic sequence
 $u_n = u_1 + (n - 1)d$
($n-1$)d, not nd
sum of n terms of an arithmetic sequence
 $S_n = \frac{n}{2}(2u_1 + (n - 1)d); \quad S_n = \frac{n}{2}(u_1 + u_n)$
second form if last term known
 n th term of a geometric sequence
 $u_n = u_1 r^{n-1}$
sum of n terms of a finite geometric sequence
 $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$
use the $r > 1$ form to avoid negatives

Compound interest
 $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$
 k = compounding periods per year
Exponents and logarithms
 $a^x = b \iff x = \log_a b$
a log undoes a power

Exponents and logarithms
 $\log_a xy = \log_a x + \log_a y$
 $\log_a \frac{x}{y} = \log_a x - \log_a y$
 $\log_a x^m = m \log_a x$
no rule for $\log(x+y)$

$\log_a x = \frac{\log_b x}{\log_b a}$
change of base for the calculator
sum of an infinite geometric sequence
 $S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$
always state $|r| < 1$

Binomial theorem $n \in \mathbb{N}$
 $(a + b)^n = a^n + {}^nC_1 a^{n-1} b + \dots + {}^nC_r a^{n-r} b^r + \dots + b^n$
general term ${}^nC_r a^{n-r} b^r$

${}^nC_r = \frac{n!}{r!(n - r)!}$
Combinations HL
 ${}^nC_r = \frac{n!}{r!(n - r)!}$
Permutations HL
 ${}^nP_r = \frac{n!}{(n - r)!}$
order matters

Extension of binomial theorem, $n \in \mathbb{Q}$
HL
 $(a + b)^n = a^n \left(1 + n \left(\frac{b}{a}\right) + \frac{n(n - 1)}{2!} \left(\frac{b}{a}\right)^2 + \dots\right)$
valid for $|b/a| < 1$

Complex numbers HL
 $z = a + bi$
Modulus-argument (polar) and exponential (Euler) form HL
 $z = r(\cos \theta + i \sin \theta) = re^{i\theta} = r \operatorname{cis} \theta$
sketch to fix the quadrant

De Moivre's theorem HL
 $[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta) = r^n \operatorname{cis} n\theta$
power: r^n and $n\theta$

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Laws of exponents
 $a^m \times a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn}, \quad a^{-n} = \frac{1}{a^n}$
Modulus and argument (HL)
 $|z| = \sqrt{a^2 + b^2}, \quad \tan \theta = \frac{b}{a}$
Complex conjugate (HL)
 $z^* = a - bi, \quad zz^* = |z|^2$
 n th roots of a complex number (HL)
 $z^{1/n} = r^{1/n} \operatorname{cis} \left(\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, \dots, n - 1$

Functions

Equations of a straight line
 $y = mx + c; \quad ax + by + d = 0; \quad y - y_1 = m(x - x_1)$
perpendicular: $m_1 m_2 = -1$
Gradient formula
 $m = \frac{y_2 - y_1}{x_2 - x_1}$
Axis of symmetry of the graph of a quadratic function
 $f(x) = ax^2 + bx + c \implies$
axis of symmetry is $x = -\frac{b}{2a}$
vertex x -coordinate

Solutions of a quadratic equation
 $ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$
Discriminant
 $\Delta = b^2 - 4ac$
 $\Delta < 0$: no real roots

Exponential and logarithmic functions
 $a^x = e^{x \ln a}; \quad \log_a a^x = x = a^{\log_a x}$
turns any base into e
Sum and product of the roots of polynomial equations HL
Sum is $-\frac{a_{n-1}}{a_n}$; product is $\frac{(-1)^n a_0}{a_n}$
watch $(-1)^n$ in the product

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Vertex form of a quadratic
 $f(x) = a(x - h)^2 + k$
Transformations
 $y = af(b(x - c)) + d$
Inverse function
 $f(f^{-1}(x)) = x$

Geometry and trigonometry

Distance between two points
 (x_1, y_1, z_1) and (x_2, y_2, z_2)
 $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Midpoint of a line segment
 $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Volume of a right-pyramid
 $V = \frac{1}{3}Ah$
Volume of a right cone
 $V = \frac{1}{3}\pi r^2 h$
Area of the curved surface of a cone
 $A = \pi rl$
 l = slant height

Volume of a sphere
 $V = \frac{4}{3}\pi r^3$
Surface area of a sphere
 $A = 4\pi r^2$
Sine rule
 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
check the ambiguous case

Cosine rule
 $c^2 = a^2 + b^2 - 2ab \cos C; \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$
three sides, or SAS
Area of a triangle
 $A = \frac{1}{2}ab \sin C$
 C is between a and b

Length of an arc (radians)
 $l = r\theta$
radians only
Area of a sector (radians)
 $A = \frac{1}{2}r^2 \theta$

Identity for $\tan \theta$
 $\tan \theta = \frac{\sin \theta}{\cos \theta}$
Pythagorean identity
 $\cos^2 \theta + \sin^2 \theta = 1$
Double angle identities
 $\sin 2\theta = 2 \sin \theta \cos \theta$
 $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
pick the form with one trig function

Reciprocal trigonometric identities HL
 $\sec \theta = \frac{1}{\cos \theta}$
 $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$
Pythagorean identities HL
 $1 + \tan^2 \theta = \sec^2 \theta$
 $\sin^2 + \cos^2 = 1 + \cos^2$
 $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$
Compound angle identities HL
 $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
 $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
note the $\mp$
 $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double angle identity for $\tan$ HL
 $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
Magnitude of a vector HL
 $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
Scalar product HL
 $v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$
 $= 0$ means perpendicular
 $v \cdot w = |v||w| \cos \theta$
Angle between two vectors HL
 $\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{|v||w|}$
lines: use direction vectors

Vector equation of a line HL
 $r = a + \lambda b$
Parametric form of the equation of a line HL
 $x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n$
Cartesian equations of a line HL
 $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
Vector product: HL
 $v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$

$\perp$ to both: a normal
 $|v \times w| = |v||w| \sin \theta$
Area of a parallelogram HL
 $A = |v \times w|$
triangle: half of this
Vector equation of a plane HL
 $r = a + \lambda b + \mu c$
Equation of a plane (using the normal vector) HL
 $r \cdot n = a \cdot n$
 n is the normal
Cartesian equation of a plane HL
 $ax + by + cz = d$
(a, b, c) is a normal

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Exact values
 $\sin 30^\circ = \frac{1}{2}, \cos 30^\circ = \frac{\sqrt{3}}{2}, \sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}, \tan 60^\circ = \sqrt{3}$
Power-reducing forms
 $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$
Angle between a line and a plane (HL)
 $\sin \theta = \frac{|b \cdot n|}{|b||n|}$

Statistics and probability

Interquartile range
 $IQR = Q_3 - Q_1$
outlier: $1.5 \times IQR$ beyond Q_1, Q_3
Mean, $\bar{x}$, of a set of data
 $\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}$
Probability of an event A
 $P(A) = \frac{n(A)}{n(U)}$
Complementary events
 $P(A) + P(A') = 1$
at least one = $1 - P(\text{none})$
Combined events
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Mutually exclusive events
 $P(A \cup B) = P(A) + P(B)$

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

given B: divide by P(B)

Independent events

$$P(A \cap B) = P(A)P(B)$$

use it to test independence

Expected value of a discrete random variable X

$$E(X) = \sum x P(X = x)$$

fair game: $E = 0$

Binomial distribution $X \sim B(n, p)$

Mean

$$E(X) = np$$

Variance

$$\text{Var}(X) = np(1 - p)$$

SD = $\sqrt{\text{variance}}$

Standardized normal variable

$$z = \frac{x - \mu}{\sigma}$$

inverse normal for unknown μ , σ

Bayes' theorem **HL**

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(B')P(A|B')}$$

draw the tree first

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

Variance σ^2 **HL**

$$\sigma^2 = \frac{\sum_{i=1}^k f_i(x_i - \mu)^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \mu^2$$

Standard deviation σ **HL**

$$\sigma = \sqrt{\frac{\sum_{i=1}^k f_i(x_i - \mu)^2}{n}}$$

Linear transformation of a single random variable **HL**

$$E(aX + b) = aE(X) + b$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

+b shifts, doesn't spread

Expected value of a continuous

random variable X **HL**

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

Variance **HL**

$$\text{Var}(X) = E[(X - \mu)^2] = E(X^2) - [E(X)]^2$$

$E(X^2)$ minus mean²

Variance of a discrete random variable

X **HL**

$$\text{Var}(X) = \sum (x - \mu)^2 P(X = x) = \sum x^2 P(X = x) - \mu^2$$

Variance of a continuous random

variable X **HL**

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

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Probabilities sum to 1

$$\sum P(X = x) = 1$$

Probability density function (HL)

$$\int_{-\infty}^{\infty} f(x) dx = 1, \quad P(a < X < b) = \int_a^b f(x) dx$$

Calculus

Derivative of x^n

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$$

roots and fractions $\rightarrow$ powers

Integral of x^n

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

never drop +C

Area between a curve $y = f(x)$ and the x -axis,

$$A = \int_a^b y dx$$

Derivative of $\sin x$

$$f(x) = \sin x \Rightarrow f'(x) = \cos x$$

Derivative of $\cos x$

$$f(x) = \cos x \Rightarrow f'(x) = -\sin x$$

Derivative of e^x

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

Derivative of $\ln x$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

Chain rule

$$y = g(u), \text{ where } u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

outer' $\times$ inner'

Product rule

$$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

Quotient rule

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

v-u' comes first

Acceleration

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Distance travelled from t_1 to t_2

$$\text{distance} = \int_{t_1}^{t_2} |v(t)| dt$$

distance uses $|v|$

Displacement from t_1 to t_2

$$\text{displacement} = \int_{t_1}^{t_2} v(t) dt$$

Standard integrals

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int e^x dx = e^x + C$$

Area of region enclosed by a curve and x -axis

$$A = \int_a^b |y| dx$$

split at the roots

Derivative of $f(x)$ from first principles

HL

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) =$$

$$\lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

write lim on every line

Standard derivatives: $\tan x$ **HL**

$$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$

$\sec x$ **HL**

$$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$$

$\text{cosec } x$ **HL**

$$f(x) = \text{cosec } x \Rightarrow f'(x) = -\text{cosec } x \cot x$$

$\cot x$ **HL**

$$f(x) = \cot x \Rightarrow f'(x) = -\text{cosec}^2 x$$

a^x **HL**

$$f(x) = a^x \Rightarrow f'(x) = a^x (\ln a)$$

$\log_a x$ **HL**

$$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$$

$\arcsin x$ **HL**

$$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$\arccos x$ **HL**

$$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$$

$\arctan x$ **HL**

$$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$$

Standard integrals **HL**

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \left(\frac{x}{a} \right) + C$$

complete the square first

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \left(\frac{x}{a} \right) + C, \quad |x| < a$$

Integration by parts **HL**

$$\int u \frac{dv}{dx} dx = uv -$$

$$\int v \frac{du}{dx} dx \quad \text{or} \quad \int u dv = uv -$$

$$\int v du$$

u = the part that simplifies

Area of region enclosed by a curve and y -axis **HL**

$$A = \int_a^b |x| dy$$

Volume of revolution about the x or y -axes **HL**

$$V = \int_a^b \pi y^2 dx \quad \text{or} \quad V = \int_a^b \pi x^2 dy$$

square, then integrate

Euler's method **HL**

$$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h$$

Integrating factor for $y' + P(x)y =$

$$Q(x) \quad \text{HL}$$

$$e^{\int P(x) dx}$$

make y'' 's coefficient 1 first

Maclaurin series **HL**

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

Maclaurin series for special functions **HL**

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

substitute, don't re-derive

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

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Tangent and normal at $x = a$

$$y - f(a) = f'(a)(x - a)$$

$$a), \quad m_{\text{normal}} = -\frac{1}{f'(a)}$$

Stationary points

$$f'(x) = 0; \quad f''(x) >$$

0 minimum, $f''(x) < 0$ maximum

Linear substitution

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C$$

Implicit differentiation (HL)

$$\frac{d}{dx} (y^n) = ny^{n-1} \frac{dy}{dx}$$

L'Hôpital's rule (HL)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Separating variables (HL)

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)} dy = \int f(x) dx$$