

Number and algebra

n th term of an arithmetic sequence
 $u_n = u_1 + (n - 1)d$
($n-1$)d, not nd
sum of n terms of an arithmetic sequence
 $S_n = \frac{n}{2}(2u_1 + (n - 1)d)$; $S_n = \frac{n}{2}(u_1 + u_n)$
second form if last term known
 n th term of a geometric sequence
 $u_n = u_1 r^{n-1}$
sum of n terms of a finite geometric sequence
 $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$, $r \neq 1$
use the $r>1$ form to avoid negatives
sum of an infinite geometric sequence
HL

$S_\infty = \frac{u_1}{1 - r}$, $|r| < 1$
always state $|r| < 1$
Compound interest
 $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$
 k = compounding periods per year
Exponents and logarithms
 $a^x = b \iff x = \log_a b$
a log undoes a power
Laws of logarithms HL
 $\log_a xy = \log_a x + \log_a y$
 $\log_a \frac{x}{y} = \log_a x - \log_a y$
 $\log_a x^m = m \log_a x$
no rule for $\log(x+y)$
Percentage error
 $\epsilon = \left| \frac{v_A - v_E}{v_E} \right| \times 100\%$
divide by the exact value
Complex numbers HL
 $z = a + bi$
Discriminant HL
 $\Delta = b^2 - 4ac$
 $\Delta < 0$: complex roots
Modulus-argument (polar) and exponential (Euler) form HL
 $z = r(\cos \theta + i \sin \theta) = re^{i\theta} = r \operatorname{cis} \theta$
multiply: $\times$ moduli, $+$ arguments
Determinant of a 2×2 matrix HL
 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$
Inverse of a 2×2 matrix HL
 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$
no inverse if $\det = 0$
Power formula for a matrix HL
 $M^n = P D^n P^{-1}$
 P 's columns match D 's order

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Laws of exponents
 $a^m \times a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$
Depreciation
 $FV = PV \left(1 - \frac{r}{100}\right)^n$
Eigenvalues (HL)
 $\det(M - \lambda I) = 0$

Functions

Solutions of a quadratic equation HL
The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$
Equations of a straight line
 $y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
perpendicular: $m_1 m_2 = -1$
Gradient formula
 $m = \frac{y_2 - y_1}{x_2 - x_1}$

Axis of symmetry of the graph of a quadratic function
 $f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$
vertex x-coordinate
Logistic function HL
 $f(x) = \frac{L}{1 + Ce^{-kx}}$, $L, k, C > 0$
 L = carrying capacity
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Gradients of perpendicular lines
 $m_1 m_2 = -1$
Exponential model
 $f(x) = ka^x + c$
Direct and inverse variation
 $y = kx^n$, $y = \frac{k}{x^n}$

Geometry and trigonometry

Area of a parallelogram
 $A = bh$
Area of a triangle
 $A = \frac{1}{2}(bh)$
Area of a trapezoid
 $A = \frac{1}{2}(a + b)h$
Area of a circle
 $A = \pi r^2$
Circumference of a circle
 $C = 2\pi r$
Volume of a cuboid
 $V = lwh$
Volume of a cylinder
 $V = \pi r^2 h$
Volume of a prism
 $V = Ah$
Area of the curved surface of a cylinder
 $A = 2\pi rh$
Distance between two points (x_1, y_1) and (x_2, y_2)
 $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Midpoint of a line segment
 $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)
 $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Midpoint of a line segment
 $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
Volume of a right-pyramid
 $V = \frac{1}{3}Ah$
Volume of a right cone
 $V = \frac{1}{3}\pi r^2 h$
Area of the curved surface of a cone
 $A = \pi rl$
 l = slant height
Volume of a sphere
 $V = \frac{4}{3}\pi r^3$
Surface area of a sphere
 $A = 4\pi r^2$
Sine rule
 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
check the ambiguous case
Cosine rule
 $c^2 = a^2 + b^2 - 2ab \cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
three sides, or SAS
Area of a triangle
 $A = \frac{1}{2}ab \sin C$
 C is between a and b
Length of an arc (degrees)
 $l = \frac{\theta}{360} \times 2\pi r$
degrees (radians in 3.7)
Area of a sector (degrees)
 $A = \frac{\theta}{360} \times \pi r^2$
Length of an arc (radians) HL
 $l = r\theta$
Area of a sector (radians) HL
 $A = \frac{1}{2}r^2\theta$

Identities HL
 $\cos^2 \theta + \sin^2 \theta = 1$
 $\tan \theta = \frac{\sin \theta}{\cos \theta}$
Transformation matrices: reflection in the line $y = (\tan \theta)x$ HL
 $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
 θ = line's angle to x-axis
horizontal stretch / stretch parallel to x -axis with a scale factor of k HL
 $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
vertical stretch / stretch parallel to y -axis with a scale factor of k HL
 $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
enlargement, with a scale factor of k , centre $(0, 0)$ HL
 $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
anticlockwise/counter-clockwise rotation of angle θ about the origin ($\theta > 0$) HL
 $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$
first transformation on the right
clockwise rotation of angle θ about the origin ($\theta > 0$) HL
 $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$
Magnitude of a vector HL
 $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
Vector equation of a line HL
 $r = a + \lambda b$
Parametric form of the equation of a line HL
 $x = x_0 + \lambda l$, $y = y_0 + \lambda m$, $z = z_0 + \lambda n$
Scalar product HL
 $v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$
= 0 means perpendicular
 $v \cdot w = |v||w| \cos \theta$
Angle between two vectors HL
 $\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{|v||w|}$
lines: use direction vectors
Vector product: HL
 $v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$
 $\perp$ to both: a normal
 $|v \times w| = |v||w| \sin \theta$
Area of a parallelogram HL
 $A = |v \times w|$
triangle: half of this

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Gradient of a perpendicular bisector
 $m_\perp = -\frac{1}{m}$
Bearings (clockwise from north)
045°, 120°, 300°

Statistics and probability

Interquartile range
 $IQR = Q_3 - Q_1$
outlier: $1.5 \times IQR$ beyond Q_n , Q_3
Mean, $\bar{x}$, of a set of data
 $\bar{x} = \frac{\sum_{i=1}^n f_i x_i}{n}$
Probability of an event A
 $P(A) = \frac{n(A)}{n(U)}$
Complementary events
 $P(A) + P(A') = 1$
at least one = $1 - P(\text{none})$
Combined events
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Mutually exclusive events
 $P(A \cup B) = P(A) + P(B)$

Conditional probability
 $P(A|B) = \frac{P(A \cap B)}{P(B)}$
given B: divide by $P(B)$
Independent events
 $P(A \cap B) = P(A)P(B)$
use it to test independence
Expected value of a discrete random variable X
 $E(X) = \sum_{i=1}^k x_i P(X = x_i)$
fair game: $E = 0$
Binomial distribution $X \sim B(n, p)$
Mean
 $E(X) = np$
Variance
 $\operatorname{Var}(X) = np(1 - p)$
 $SD = \sqrt{\text{variance}}$
Linear transformation of a single random variable HL
 $E(aX + b) = aE(X) + b$
 $\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$
+b shifts, doesn't spread
Linear combinations of n independent random variables, $X_1, X_2, \dots, X_n$ HL
 $E(a_1 X_1 \pm a_2 X_2 \pm \dots \pm a_n X_n) = a_1 E(X_1) \pm a_2 E(X_2) \pm \dots \pm a_n E(X_n)$
 $\operatorname{Var}(a_1 X_1 \pm a_2 X_2 \pm \dots \pm a_n X_n) = a_1^2 \operatorname{Var}(X_1) + a_2^2 \operatorname{Var}(X_2) + \dots + a_n^2 \operatorname{Var}(X_n)$
variances always add
Sample statistics. Unbiased estimate of population variance s_{n-1}^2 HL
 $s_{n-1}^2 = \frac{n}{n-1} s_n^2$
unbiased: $n - 1$
Poisson distribution $X \sim \operatorname{Po}(m)$ HL
Mean HL
 $E(X) = m$
Variance HL
 $\operatorname{Var}(X) = m$
mean $\approx$ variance $\rightarrow$ Poisson
Transition matrices HL
 $T^n s_0 = s_n$
steady state: $Ts = s$

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Standardized normal variable
 $z = \frac{x - \mu}{\sigma}$
Expected frequency (χ^2 test)
expected = $\frac{\text{row total} \times \text{column total}}{\text{grand total}}$
Degrees of freedom (χ^2 test for independence)
 $\nu = (r - 1)(c - 1)$
Outliers
 $x < Q_1 - 1.5 \times IQR$ or $x > Q_3 + 1.5 \times IQR$

Calculus

Derivative of x^n
 $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
roots and fractions $\rightarrow$ powers
Derivative of $\sin x$ HL
 $f(x) = \sin x \Rightarrow f'(x) = \cos x$
Derivative of $\cos x$ HL
 $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
Derivative of $\tan x$ HL
 $f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$
Derivative of e^x HL
 $f(x) = e^x \Rightarrow f'(x) = e^x$
Derivative of $\ln x$ HL
 $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
Chain rule HL
 $y = g(u)$, where $u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
outer' $\times$ inner'
Product rule HL
 $y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient rule **HL**

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

v-u' comes first

Integral of x^n

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

never drop +C

trapezoidal rule

$$\int_a^b y \, dx \approx \frac{1}{2}h((y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}))$$

n strips, n+1 values

Standard integrals **HL**

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x + C$$

$$\int e^x \, dx = e^x + C$$

Area of region enclosed by a curve and x or y -axes **HL**

$$A = \int_a^b |y| \, dx \quad \text{or} \quad A = \int_a^b |x| \, dy$$

Volume of revolution about x or y -axes **HL**

$$V = \int_a^b \pi y^2 \, dx \quad \text{or} \quad V = \int_a^b \pi x^2 \, dy$$

square, then integrate

Acceleration **HL**

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \frac{dv}{ds}$$

use when a is given in s

Distance travelled from t_1 to t_2 **HL**

$$\text{distance} = \int_{t_1}^{t_2} |v(t)| \, dt$$

distance uses |v|

Displacement from t_1 to t_2 **HL**

$$\text{displacement} = \int_{t_1}^{t_2} v(t) \, dt$$

Euler's method **HL**

$$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h$$

smaller h, better estimate

Euler's method for coupled systems

HL

$$x_{n+1} = x_n + h \times$$

$$f_1(x_n, y_n, t_n), \quad y_{n+1} = y_n + h \times$$

$$f_2(x_n, y_n, t_n), \quad t_{n+1} = t_n + h$$

update x and y together

Exact solution for coupled linear differential equations **HL**

$$\mathbf{x} = A e^{\lambda_1 t} \mathbf{p}_1 + B e^{\lambda_2 t} \mathbf{p}_2$$

λ : node, saddle or spiral

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Tangent and normal at $x = a$

$$y - f(a) = f'(a)(x -$$

$$a), \quad m_{\text{normal}} = -\frac{1}{f'(a)}$$

Stationary points

$$f'(x) = 0$$

Separating variables (HL)

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)} dy = \int f(x) dx$$

Phase portraits (HL)

λ_1, λ_2 real, same sign: node; opposite signs: saddle; complex: spiral

Second-order equations (HL)

$$\ddot{x} + a\dot{x} + bx = 0 \Rightarrow \dot{x} = y, \quad \dot{y} = -ay - bx$$